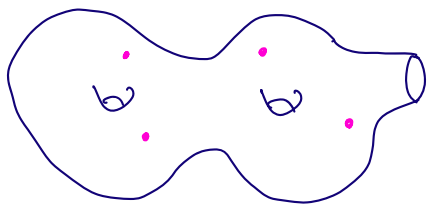


M : oriented, open manifold.

$$q = \dim M \geq 2.$$

(Unordered) Configuration spaces:

$$C_n(M) := \{ (x_1, \dots, x_n) \mid x_i \in M, x_i \neq x_j \} / S_n$$



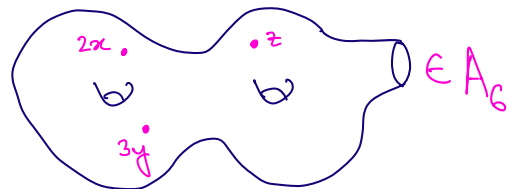
$C_n(M)$: manifold of dim nq

Symmetric Products:

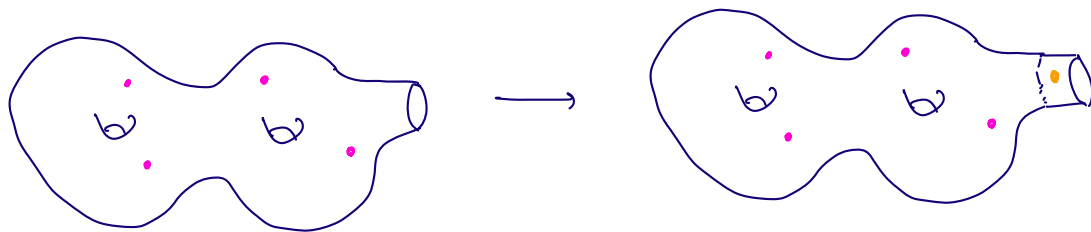
$$A_n(M) := \{ (x_1, \dots, x_n) : x_i \in M \} / S_n \quad (\text{So the } x_i \text{ need not be distinct!})$$

$A_n(M)$: • not necessarily a manifold

$$\bullet H_c^i(A_n(M)) = 0 \quad \text{for } i > nq$$



Stabilization Map: $C_n(M) \rightarrow C_{n+1}(M)$



Goal: $C_n(M) \rightarrow C_{n+1}(M)$ is a H_* -equiv for $* \leq \lfloor \frac{n}{2} \rfloor$

Equivalently: $\mathbb{R}^2 \times C_n(M) \xrightarrow{\text{open}} C_{n+1}(M)$ is a H_* -equiv for $* \leq \lfloor \frac{n}{2} \rfloor$

(+) Dual Goal: $\mathbb{R}^2 \times C_n(M) \xrightarrow{\text{open}} C_{n+1}(M)$ is a H_c^* -equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$
 ↑ will let us use LES for $U \hookrightarrow X$ in H_c^*

This talk : Prove (+) assuming the analogous statement for symmetric products $A_n(M)$.

(will state this explicitly in a moment)

Rmk : • $C_n(M)$ is not a homotopy invariant, but $A_n(M) = M^n/S_n$ is
(eg: $C_2(\mathbb{R}^d) \simeq S^{d-1}$)

• $A_n(\mathbb{C}) \cong \mathbb{C}^n$ sym. products more accessible than $C_n(M)$
(Warning : $H_c^*(A_n)$ is NOT htpy-invariant!)

Notation : Write $\{x, x, y, y, y, z\} \in A_6(M)$ as $2x + 3y + z$

We can write any $\xi \in A_n(M)$ (non-uniquely) as $\xi = 2T + U$
 $T \in A_{n-2k}$ $U \in A_k$

$$\begin{aligned} \text{Eq: } \{x, x, y, y, y, z\} &= 2 \underset{T}{(x)} + \underset{U}{(3y + z)} \\ &= 2 \underset{T}{(x + y)} + \underset{U}{(y + z)} \end{aligned}$$

Define $A_{n,k} := \{ \xi = 2T + U \in A_n \mid |T| \geq k \}$

$$\begin{aligned} \text{Eq: } \{x, x, y, y, y, z\} &= 2 \underset{T}{(x)} + \underset{U}{(3y + z)} \in A_{n,1} \\ &= 2 \underset{T}{(x + y)} + \underset{U}{(y + z)} \in A_{n,2} \subset A_{n,1} \end{aligned}$$

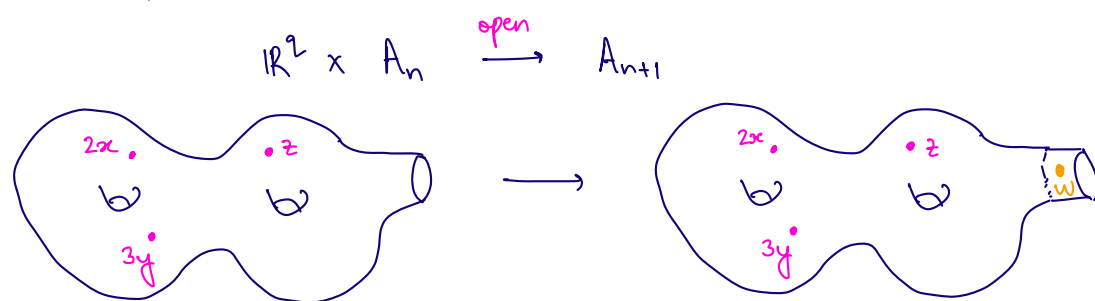
Note : $A_n = A_{n,0} \supset A_{n,1} \supset A_{n,2} \supset \dots \supset A_{n, \lfloor \frac{n}{2} \rfloor} \supset A_{n, \lfloor \frac{n}{2} \rfloor + 1} = \emptyset$

"Arnol'd - Segal Filtration"

$$A_{n,k} - A_{n,k+1} = A_k \times C_{n-2k}$$

] basis for relating stability statement for C_n to that of A_n

Stabilization map for A_n :



This is filtered: $\mathbb{R}^2 \times A_{n,k} \hookrightarrow A_{n+1,k}$

LES for H_c^* : $A_{n,k} - A_{n,k+1} = A_k \times C_{n-2k}$

$$A_k \times C_{n-2k} \xrightarrow{\text{open}} A_{n,k}$$

$$\mathbb{R}^2 \times A_{n,k} - \mathbb{R}^2 \times A_{n,k+1} = \mathbb{R}^2 \times A_k \times C_{n-2k}$$

Get :

$$\begin{array}{ccccccc} \dots \leftarrow H_c^i(\mathbb{R}^2 \times A_{n,k+1}) & \leftarrow & H_c^i(\mathbb{R}^2 \times A_{n,k}) & \leftarrow & H_c^i(A_k \times (\mathbb{R}^2 \times C_{n-2k})) & \leftarrow & H_c^{i-1}(\mathbb{R}^2 \times A_{n,k+1}) \leftarrow \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots \leftarrow H_c^i(A_{n+1,k+1}) & \leftarrow & H_c^i(A_{n+1,k}) & \leftarrow & H_c^i(A_k \times C_{n-2k+1}) & \leftarrow & H_c^{i-1}(A_{n+1,k+1}) \leftarrow \dots \end{array}$$

Recall our goal :

Assuming that

$$\mathbb{R}^2 \times A_n \rightarrow A_{n+1} \text{ is a } H_c^* \text{-equiv for } * \geq (n+1)q - \lfloor \frac{n}{2} \rfloor \quad \forall n \geq 1 \quad (1)$$

Prove that

$$\mathbb{R}^2 \times C_n \rightarrow C_{n+1} \text{ is a } H_c^* \text{-equiv for } * \geq (n+1)q - \lfloor \frac{n}{2} \rfloor \quad \forall n \geq 1 \quad (2)$$

Note: (2) holds for $n = 1$

$$\mathbb{R}^2 \times C_1(M) \cong \mathbb{R}^2 \times M, \dim 2q$$

$$(n+1)q - \lfloor \frac{n}{2} \rfloor = 2q - 0 = 2q$$

So we'll induct on n .

Lemma: for $k > 0$, $\mathbb{R}^2 \times A_{n,k} \rightarrow A_{n+1,k}$ is a H_c^* -equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

Pf: True for $k > \lfloor \frac{n+1}{2} \rfloor$, since in that case $A_{n,k} = \emptyset = A_{n+1,k}$.

So will downwards induct on k .

$$\begin{array}{ccccccc} \dots \leftarrow H_c^i(\mathbb{R}^2 \times A_{n,k+1}) & \leftarrow & H_c^i(\mathbb{R}^2 \times A_{n,k}) & \leftarrow & H_c^i(A_k \times (\mathbb{R}^2 \times C_{n-2k})) & \leftarrow & H_c^{i-1}(\mathbb{R}^2 \times A_{n,k+1}) \leftarrow \dots \\ \downarrow (*) & & \downarrow \text{Want } \cong & & \downarrow (*) & & \downarrow (***) \\ \dots \leftarrow H_c^i(A_{n+1,k+1}) & \leftarrow & H_c^i(A_{n+1,k}) & \leftarrow & H_c^i(A_k \times C_{n-2k+1}) & \leftarrow & H_c^{i-1}(A_{n+1,k+1}) \leftarrow \dots \end{array}$$

(*) : $\cong$ by induction on n , since $n-2k < n$

(**), (***) : $\cong$ by (downwards) induction on k

$\Rightarrow \cong$ in desired range

Caveat: We actually used the fact that

$$A_k \times \mathbb{R}^2 \times C_{n-2k} \rightarrow A_k \times C_{n-2k+1}$$

is H_C^* - iso for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

instead of

$$\mathbb{R}^2 \times C_{n-2k} \rightarrow C_{n-2k+1}$$

This can be easily shown using the Künneth formula - will postpone to end.

for now, we have:

Lemma: for $k > 0$, $\mathbb{R}^2 \times A_{n,k} \rightarrow A_{n+1,k}$ is a H_C^* - equiv for $* \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$

set $k=0$ in LES (and recall $A_n = A_{n,0}$):

$$\begin{array}{ccccccc} \dots \leftarrow H_C^i(\mathbb{R}^2 \times A_{n,1}) & \leftarrow & H_C^i(\mathbb{R}^2 \times A_n) & \leftarrow & H_C^i(A_0 \times (\mathbb{R}^2 \times C_n)) & \leftarrow & H_C^{i-1}(\mathbb{R}^2 \times A_{n,1}) \leftarrow \dots \\ & \downarrow * & & \downarrow (2) & & \downarrow (1) & & \downarrow ** \\ \dots \leftarrow H_C^i(A_{n+1,1}) & \leftarrow & H_C^i(A_{n+1}) & \leftarrow & H_C^i(A_0 \times C_{n+1}) & \leftarrow & H_C^{i-1}(A_{n+1,1}) \leftarrow \dots \end{array}$$

$*$, $**$: H_C^* - equiv in desired range by Lemma

(2) : H_C^* - equiv in desired range by assumption

$\Rightarrow (1)$: H_C^* - equiv in desired range

Künneth Claim :

Suppose $Y \rightarrow Z$ is a H_c^* -equiv for $* \geq m$
and $H_c^*(X) = 0$ for $* > d$.

Then $X \times Y \rightarrow X \times Z$ is a H_c^* -equiv
for $* \geq d + m$.

Pf: Künneth : $H_c^i(X \times Y) \cong \bigoplus_{a+b=i} H_c^a(X) \otimes H_c^b(Y)$

Can assume $a \leq d$ since $H_c^a(X) = 0$ o/w.

If $i \geq d + m$, this forces $b = i - a \geq m$.

Apply to : $X = A_k$
 $Y = \mathbb{R}^q \times C_{n-2k}$, $Z = C_{n-2k+1}$

$$\leadsto A_k \times \mathbb{R}^q \times C_{n-2k} \rightarrow A_k \times C_{n-2k+1}$$

is H_c^* -equiv for $* \geq (kq) + ((n-2k+1)q - \lfloor \frac{n-2k}{2} \rfloor)$

$$= \underbrace{(n+1)q - kq - \lfloor \frac{n}{2} \rfloor + k}_{< (n+1)q - \lfloor \frac{n}{2} \rfloor}$$

$$\Rightarrow H_c^* \text{-equiv for } * \geq (n+1)q - \lfloor \frac{n}{2} \rfloor$$